Math 4315 PDE's

To solve

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u)$$

if $u_s = u_x x_s + u_y y_s$

if we choose

$$\begin{aligned} x_s &= a \\ y_s &= b \\ u_s &= c \end{aligned}$$

$\}$ so $u_s = a u_x + b u_y = c$

so these are ODE's (really PDE's but there are no r derivatives)

so we have functions of integration and we eliminate s then r giving our sol^n. The following illustrate

ex 1 Solve

$$u_x + y u_y = 2$$

if

$$\begin{aligned} x_s &= 1 &\Rightarrow\quad x &= s + a(r) \\ y_s &= y &\Rightarrow\quad y &= b(r)e^{s} \\ u_s &= 2 & u &= 2s + c(r) \end{aligned}$$

(1) $y = b(r)\, e^{s}$

$= b(r)\, e^{x-a(r)} = b(r)\, e^{-a(r)} e^{x}$

$= d(r)\, e^{x}$

$r = d^{-1}\left(\frac{y}{e^x}\right) = d^{-1}(y e^{-x})$

(2) $u = 2(x - a(r)) + c(r)$

$= 2x - 2a(r) + c(r)$

$= 2x + e(r)$

$u = 2x + f(y e^{-x})$

Ex 2 $x u_x + (2x - y) u_y = x$

$x_s = x \Rightarrow x = a(r)\, e^{s}$

$y_s = 2x - y$ $\quad y_s + y = 2a(r)\, e^{s}$

$u_s = x$ $\quad u_s = a(r)\, e^{s} \Rightarrow u = a(r)\, e^{s} + c(r)$

integrating factor $\mu = e^{s}$

$\frac{\partial}{\partial s} e^{s} y = 2a(r)\, e^{2s}$ $\qquad e^{s} y = a(r)\, e^{2s} + b(r)$

$y = a(r)\, e^{s} + b(r)\, e^{-s}$

so for $x = a(r) e^{s}$

$y = a(r) e^{s} + b(r) e^{-s}$

$u = a(r) e^{s} + c(r)$

Get rid of s & then r

$e^{s} = \frac{x}{a(r)}$

so $y = x + b(r) e^{-s}$

$= x + \frac{b(r) a(r)}{x}$

$xy = x^2 + b(r) a(r)$ so $r = d^{-1}(xy - x^2)$

$u = a(r) e^{s} + c(r)$

$u = x + f(xy - x^2)$ Sol^n

check $u_x = 1 + f'(\ ) \cdot (y - 2x)$

$u_y = f'(\ ) \cdot x$

LS $x u_x + (2x - y) u_y$

$= x + f'(\ ) \cdot (xy - \cancel{2x^2}) + (\cancel{2x - y}) f'(\ ) \cdot x$

$= x = $ RS. ✓

Ex 3 $\quad u_x + u u_y = 0$

cf $\quad u_s = u_x x_s + u_y y_s$

choose $\quad x_s = 1$

$y_s = u \leftarrow$ we need u here

$u_s = 0$

$x = s + a(r) \qquad y_s = c(r) \qquad y = c(r)s + b(r)$

$u = c(r)$

so $\quad \left.\begin{array}{l} x = s + a(r) \\ y = c(r)s + b(r) \\ u = c(r) \end{array}\right] \quad y = c(r)[x + a(r)] + b(r)$

$= x\,c(r) + a(r)c(r) + b(r)$

$y = xu + d(r)$

$u = c(r)$

so $\quad y - xu = f(u)$

OR $\quad u = g(y - xu) \leftarrow$ cannot solve this for u in general

Ex 4 $\quad u u_x + y u_y = u + 2y^2 \qquad u(x,1) = x+1$

cf $\quad u_s = u_x x_s + u_y y_s$

cf $\quad x_s = u \qquad\qquad u = ?$

$y_s = y \longrightarrow y = b(r) e^s$

then $\quad u_s = u + 2y^2 \qquad u_s - u = 2b^2(r) e^{2s} \qquad \mu = e^{-s}$

$\frac{\partial}{\partial s} \; e^{-s} u = 2b^2(r) e^s$

$e^{-s} u = 2b^2(r) e^s + c(r) \qquad u = 2b^2(r) e^{2s} + c(r) e^s$

$x_s = 2b^2(r) e^{2s} + c(r) e^s$

$x = b^2(r) e^{2s} + c(r) e^s + a(r)$

so far $\quad x = b^2(r) e^{2s} + c(r) e^s + a(r)$

$y = b(r) e^s$

$u = 2b^2(r) e^{2s} + c(r) e^s$

$x - y^2 = c(r) e^s + a(r) \qquad u - x - y^2 = a(r)$

$u - 2y^2 = c(r) e^s \qquad \frac{u - 2y^2}{y} = \frac{c(r)}{b(r)} = d(r)$

$y = b(r) e^s$

$$\frac{u - 2y^2}{y} = f(u - x - y^2)$$

$$x+1-2 = f(x+1-x-1)$$

$$= f(0) \; ?$$

also $u - x - y^2 = g\left(\frac{u-2y^2}{y}\right)$

$$x+1-x-1 = g(x-1)$$

$$g(x-1) = 0 \text{ so } g(x) = 0$$

so $u = x+y^2$