Math 4381 / 6378 Symmetry

So far we ~~have~~ considered the symmetries of ODEs.

If $y' = F(x, y)$ is invariant under

$$\bar{x} = x + \varepsilon X(x,y) + O(\varepsilon^2)$$
$$\bar{y} = y + \varepsilon Y(x,y) + O(\varepsilon^2)$$

then Lie's invariance condition is

$$Y_x + (Y_y - X_x) F - X_y F^2 = X F_x + Y F_y \qquad (1)$$

If we introduce the infinitesimal operator

$$\Gamma = X \frac{\partial}{\partial x} + Y \frac{\partial}{\partial y}$$

and its 1^{st} extension

$$\Gamma^{(1)} = \Gamma + Y[x] \frac{\partial}{\partial y'}$$

where $\dfrac{d\bar{y}}{d\bar{x}} = \dfrac{dy}{dx} + Y[x]\,\varepsilon + O(\varepsilon^2)$

then (1) can be written as

$$\Gamma^{(1)}\Delta\Big|_{\Delta=0} = 0$$

where $\Delta = y' - F(x,y)$

<u>Note:</u> we have a formula for $Y_{[x]}$ as

$$Y_{[x]} = D_x(Y) - y' D_x(X)$$

We then extended to 2nd order ODE.

Lie's invariance condition

$$\Gamma^{(2)}\Delta\Big|_{\Delta=0} = 0$$

where $\Delta = y'' - F(x,y,y')$

<u>Note</u>

$\bar{y}'' = y'' + Y_{[xx]}\varepsilon + O(\varepsilon^2)$

$$\Gamma^{(2)} = \Gamma^{(1)} + Y_{[xx]}\varepsilon + O(\varepsilon^2)$$

where

$$Y_{[xx]} = D_x(Y_{[x]}) - y'' D_x(X)$$

This extends to any order we wish.

Now we turn to PDE. Suppose we have some 1st order PDE

$$F(t, x, u, u_t, u_x) = 0 \qquad (2)$$

say $\quad u_t = u_x^2$

Can we find the invariance (symmetries) of this. We introduce infinitesimal trans

$$\bar{t} = t + T(t, x, u)\varepsilon + O(\varepsilon^2)$$

$$\bar{x} = x + X(t, x, u)\varepsilon + O(\varepsilon^2)$$

$$\bar{u} = u + U(t, x, u)\varepsilon + O(\varepsilon^2)$$

& extended transformations

$$\bar{u}_{\bar{t}} = u_t + U[t]\varepsilon + O(\varepsilon^2)$$

$$\bar{u}_{\bar{x}} = u_x + U[x]\varepsilon + O(\varepsilon^2)$$

invariance of (2) gives (to order ε^2)

$$T F_t + X F_x + U F_u + U[t] F_{u_t} + U[x] F_{u_x} = 0 \qquad (3)$$

where $F(t, x, u, u_t, u_x) = 0$

If we introduce the infinitesimal operator

$$\Gamma = T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + U\frac{\partial}{\partial u}$$

& 1^{st} extension

$$\Gamma^{(1)} = \Gamma + U[t]\frac{\partial}{\partial u_t} + U[x]\frac{\partial}{\partial u_x}$$

Lie invariance condition (3) can be written as

$$\Gamma^{(1)}\Delta\Big|_{\Delta=0}$$

where $\Delta = F(t, x, u, u_t, u_x)$ The same as for ODE.

Now we need formulas for $U[t]$ & $U[x]$.

So we consider (we will use Jacobians)

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial(\bar{u}, \bar{x})}{\partial(\bar{t}, \bar{x})} = \frac{\partial(\bar{u}, \bar{x})}{\partial(t, x)} \Big/ \frac{\partial(\bar{t}, \bar{x})}{\partial(t, x)}$$

$$= \begin{vmatrix} \bar{u}_t & \bar{u}_x \\ \bar{x}_t & \bar{x}_x \end{vmatrix} \Big/ \begin{vmatrix} \bar{t}_t & \bar{t}_x \\ \bar{x}_t & \bar{x}_x \end{vmatrix}$$

we define total differential operator

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + \cdots$$

so

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\begin{vmatrix} u_t + \varepsilon D_t(\overline{U}) & u_x + \varepsilon D_x(\overline{U}) \\ \varepsilon D_t(X) & 1 + \varepsilon D_x(X) \end{vmatrix}}{\begin{vmatrix} 1 + \varepsilon D_t(T) & \varepsilon D_x(T) \\ \varepsilon D_t(X) & 1 + \varepsilon D_x(X) \end{vmatrix}} + O(\varepsilon^2)$$

$$= \frac{u_t + \varepsilon\left[D_t(\overline{U}) + u_t D_x(X) - u_x D_t(X) \right]}{1 + \varepsilon\left[D_t(T) + D_x(X) \right]} + O\varepsilon^2$$

$$= u_t + \varepsilon\left[D_t(\overline{U}) + u_t D_x(X) - u_x D_t(X) \right]$$
$$* \left\{ 1 - \varepsilon\, D_t(T) + D_x(X) \right\} + O(\varepsilon^2)$$

$$= u_t + \varepsilon \left\{ \begin{array}{l} D_t(\overline{U}) + \cancel{u_t D_x(X)} - u_x D_t(X) \\ - u_t D_t(T) - \cancel{u_t D_x(X)} \end{array} \right\} + O(\varepsilon^2)$$

giving

$$\bar{u}_{\bar{t}} = u_t + \epsilon\left[D_t(\mathcal{U}) - u_t D_t(T) - u_x D_t(X)\right] + O(\epsilon^2)$$

so $\bar{\mathcal{U}}[t] = D_t(\mathcal{U}) - u_t D_t(T) - u_x D_x(X)$

~~A~~ similar calculation gives

$$\bar{\mathcal{U}}[x] = D_x(\mathcal{U}) - u_t D_x(T) - u_x D_x(X)$$

Ex Calculate the symmetries of

$$u_t = u_x^2$$

Lie's Inv. Cond.

$$\bar{\mathcal{U}}[t] = 2u_x \bar{\mathcal{U}}[x]$$

so $\mathcal{U}_t + \mathcal{U}_u u_t - u_t(T_t + T_u u_t) - u_x(X_t + X_u u_t)$

$$= 2u_x\left[\begin{array}{l} \mathcal{U}_x + \mathcal{U}_u u_x - u_t(T_x + T_u u_x) \\ \quad - u_x(X_x + X_u u_x) \end{array}\right]$$

now let $u_t = u_x^2$ & isolate coeff wrt u_x

$1)\quad \tau_t = 0$ 1a

$u_x)\quad -X_t = 2\tau_x$ 1b

$u_x^2)\quad \tau_u - T_t = 2\tau_u - 2X_x$ 1c

$u_x^3)\quad -X_u = -2T_x - 2X_u$ 1d

$u_x^4)\quad -T_u = -2T_u$ 1e

so 1(e) say $T_u = 0 \Rightarrow T = A(t,x)$

from 1d $\quad X_u = -2T_x = -2A_x$

so $\quad X = -2A_x u + B(t,x)$

From 1(c) $\quad \tau_u = 2X_x - T_t = -4A_{xx}u + 2B_x - A_t$

so $\tau = -2A_{xx}u^2 + (2B_x - A_t)u + C(t,x)$

From 1(a) $\quad \tau_t = 0$

$\Rightarrow \quad -2A_{xxt}u^2 + (2B_{tx} - A_{tt})u + C_t = 0$

$\Rightarrow \quad A_{xxt} = 0 \qquad 2B_{tx} - A_{tt} = 0 \qquad C_t = 0$ (2)

From (1b) when $2\theta_x + X_t = 0$

$$\Rightarrow -4A_{xxx}u^2 + 2(2B_{xx} - A_{tx})u + 2C_x - 2A_{tx}u + B_t = 0$$

$$\Rightarrow \quad -4A_{xxx} = 0 \qquad 3a$$

$$4B_{xx} - 4A_{tx} = 0 \qquad 3b$$

$$2C_x + B_t = 0 \qquad 3c$$

3a gives

$$A = a_1(t)x^2 + a_2(t)x + a_3(t) \qquad a_1 - a_3 \text{ arb fcts}$$

2a $\Rightarrow$ $a_1 = c_1$ arb const.

3b gives $B_{xx} = A_{tx} = a_2'(t)$

$$B = \frac{a_2'(t)}{2}x^2 + b_1(t)x + b_2(t) \qquad b_1, b_2 \text{ arb fcts}$$

3c $$C_x = -\frac{1}{2}B_t = -\frac{1}{2}\left\{\frac{a_2''}{2}x^2 + b_1'x + b_2'\right\}$$

$$C = -\frac{a_2''x^3}{12} - \frac{b_1'x^2}{4} - \frac{b_2'x}{2} + c(t)$$

c arb fct.

2(e) gives

$$2\left[a_2'' x + b_1' \right] - \left[a_2'' x + a_3'' \right] = 0$$

$$\Rightarrow \quad 2a_2'' - a_2'' = 0 \qquad 2b_1' - a_3'' = 0$$

2 f) $\quad a_2''' = 0 \qquad b_1'' = 0 \qquad b_2'' = 0 \qquad c' = 0$

we solve these giving

$$a_2 = c_2 t + c_3 \qquad\qquad a_3 = c_4 t^2 + c_9 t + c_{10}$$

$$b_1 = c_4 t + c_5$$

$$b_2 = c_6 t + c_7$$

$$c = c_8$$

giving

$$T = c_1 x^2 + (c_2 t + c_3) x + c_4 t^2 + c_9 t + c_{10}$$

$$X = -2(2c_1 x + c_2 t + c_3) u + \frac{c_2}{2} x^2 + (c_4 t + c_5) x + c_6 t + c_7$$

$$U = -4c_1 u^2 + (c_2 x - c_9 + 2c_5) u - \frac{c_4}{4} x^2 - \frac{c_6 x}{2} + c_8$$