Spring 2026 — Math 3331
Second Order ODEs - Constant Coefficient

We now consider solving

$$a(x)y'' + b(x)y' + c(x)y = 0. \tag{1}$$

where $a(x), b(x)$ and $c(x)$ are all constants, namely

$$ay'' + by' + cy = 0 \tag{2}$$

For the moment, let us consider when $a = 0$. This gives $by' + cy = 0$ or $y' = ry$ which has as its solution $y = ce^{rx}$. Motivated by this, we seek solution of (2) in the form

$$y = e^{rx} \tag{3}$$

Substituting this and derivatives into (2) gives

$$\left(ar^2 + br + c\right) e^{rx} = 0, \tag{4}$$

and since $e^{rx} \neq 0$ we have

$$ar^2 + br + c = 0, \tag{5}$$

a quadratic equations for r. This is called the *characteristic equation*. Here we present three possible cases depending on the roots of the quadratic.

Case 1 - Distinct Roots $r = r_1, r_2$. In this case the two independent solutions are

$$y_1 = e^{r_1 x}, \quad y_2 = e^{r_2 x} \tag{6}$$

and the general solution is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} \tag{7}$$

Case 2 - Repeated Roots $r = r_1, r_1$ In this case we only have one solution (we suppress the subscript 1)

$$y_1 = e^{rx}, \tag{8}$$

To find the second solution, we use reduction of order. Let $y_2 = ue^{rx}$. In doing so, we find the second solution to be $y_2 = xe^{rx}$ and the general solutions is

$$y = c_1 e^{rx} + c_2 x e^{rx} \tag{9}$$

Case 3 - Complex Roots $r = \alpha \pm \beta i$

So our solutions are of the form

$$y_1 = e^{(\alpha-\beta i)x}, \quad y_2 = e^{(\alpha+\beta i)x} \tag{10}$$

and the general solutions is

$$y = c_1 e^{(\alpha-\beta i)x} + c_2 e^{(\alpha+\beta i)x} \tag{11}$$

Now, as there is the complex i in this solution, we need to manipulate it a little (using Euler's formula $e^{ix} = \cos x + i \sin x$)

$$\begin{aligned} y &= c_1 e^{\alpha x} e^{-\beta i x} + c_2 e^{\alpha x} e^{\beta i x} \\ &= e^{\alpha x} \left(c_1 \left(\cos \beta x - I \sin \beta x \right) + c_2 \left(\cos \beta x + I \sin \beta x \right) \right) \\ &= e^{\alpha x} (c_1 + c_2) \cos \beta x + i(-c_1 + c_2) e^{\alpha x} \sin \beta x \\ &= \bar{c}_1 e^{\alpha x} \cos \beta x + \bar{c}_2 e^{\alpha x} \sin \beta x. \end{aligned} \tag{12}$$

where in the last step we set $c_1 + c_2 = \bar{c}_1$ and $i(-c_1 + c_2) = \bar{c}_2$ and we drop the bars.

The following example illustrate.

Example 1.

Find the solution to

$$y'' - 5y' + 6y = 0. \tag{13}$$

The characteristic equation is

$$r^2 - 5r + 6 = 0 \tag{14}$$

The roots are $r = 2, 3$, the two independent solutions are $y = e^{2x}$ and $y = e^{3x}$ and the general solution

$$y = c_1 e^{2x} + c_2 e^{3x} \tag{15}$$

Example 2.

Find the solution to

$$y'' + 6y' + 9y = 0. \tag{16}$$

The characteristic equations is

$$r^2 + 6r + 9 = 0 \tag{17}$$

The roots are $r = -3, -3$, the two independent solutions are $y = e^{-3x}$ and $y = xe^{-3x}$ and the general solution

$$y = c_1 e^{-3x} + c_2 x e^{-3x} \tag{18}$$

Example 3.

Find the solution to

$$y'' - 2y' + 5y = 0. \tag{19}$$

The characteristic equations is

$$r^2 - 2r + 5 = 0 \tag{20}$$

Using the quadratic formula we obtain

$$r = \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i \tag{21}$$

Here $\alpha = 1$ and $\beta = 2$ and the general solution

$$y = c_1 e^x \cos(2x) + c_2 e^x \sin(2x) \tag{22}$$

Example 4.

Find the solution to

$$y'' - 4y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = 1 \tag{23}$$

The characteristic equations is

$$r^2 - 4r + 4 = 0 \tag{24}$$

The roots are $r = 2, 2$ and the general solution

$$y = c_1 e^{2x} + c_2 x e^{2x} \tag{25}$$

Now we impose the BC so

$$y(0) = 2 \text{ gives } c_1 = 2 \tag{26}$$

For the second BC we need y' so

$$y' = (2c_1 + c_2)e^{2x} + 2c_2 x e^{2x} \tag{27}$$

and

$$y'(0) = 1 \text{ gives } 2c_1 + c_2 = 1 \quad \rightarrow \quad c_2 = -3 \tag{28}$$

and so the general solution is

$$y = 2e^{2x} - 3xe^{2x} \tag{29}$$