Math 4381 / 6378 Symmetry

So far we have been using Lie Groups to simplify 1st order ODE. For example

$$\frac{dy}{dx} = 2y^2 + xy^3 \qquad (1)$$

is invariant under

$$\bar{x} = e^{\varepsilon} x, \quad \bar{y} = e^{-\varepsilon} y \qquad (2)$$

To this we associate an infinitesimal Lie group where

$$\bar{x} = x + X(x,y)\,\varepsilon + O(\varepsilon^2)$$

$$\bar{y} = y + Y(x,y)\,\varepsilon + O(\varepsilon^2)$$

where $X(x,y) = \left.\frac{\partial \bar{x}}{\partial \varepsilon}\right|_{\varepsilon=0}$, $Y(x,y) = \left.\frac{\partial \bar{y}}{\partial \varepsilon}\right|_{\varepsilon=0}$

& here (2) becomes

$$X = x, \quad Y = -y.$$

In order to find the transformation leading to a separable eq^n we solve

$$X r_x + Y r_y = 0 \qquad X s_x + Y s_y = 1$$

For our example these are

$$x r_x - y r_y = 0 \qquad x s_x - y s_y = 1$$

These solve to give

$$r = R(xy) \qquad s = \ln x + \tilde{S}(xy)$$

we choose $r = xy \quad s = \ln x$

or $\quad x = e^{s} \quad y = r e^{-s}$

Now $\quad \dfrac{dy}{dx} = \dfrac{e^{-s} - r e^{-s} s'}{e^{s} s'} = e^{-2s}\left(\dfrac{1 - r s'}{s'}\right)$

sub into (1) $\quad e^{-2s}\left(\dfrac{1-rs'}{s'}\right) = 2r^2 e^{-2s} + e^{s} r^3 e^{-3s}$

$$\frac{1 - rs'}{s'} = 2r^2 + r^3 \Rightarrow s' = \frac{1}{r(r+1)^2} \quad \text{separable}$$

So once we know the Lie group we can find the infinitesimal which leads to the transformation.

If $\bar{x} = f(x,y,\varepsilon)$ $\bar{y} = g(x,y,\varepsilon)$

$$\frac{d\bar{y}}{d\bar{x}} = \frac{g_x + g_y y'}{f_x + f_y y'}$$

$$\frac{d\bar{y}}{d\bar{x}} = F(\bar{x},\bar{y}) \Rightarrow \frac{g_x + g_y y'}{f_x + f_y y'} = F(f,g)$$

& if we want $y' = F$

then $y' = \dfrac{f_x F - g_x}{g_y - f_y F} = F(x,y)$ ← way to load

Besides, what we really want are the infinitesimals.

Now we seek invariance of

$$\frac{dy}{dx} = F(x,y)$$

under
$$\bar{x} = x + \varepsilon X(x,y) + O(\varepsilon^2)$$
$$\bar{y} = y + \varepsilon Y(x,y) + O(\varepsilon^2)$$

so
$$\frac{d\bar{y}}{d\bar{x}} = \frac{y' + \varepsilon\left[Y_x + Y_y y'\right] + O(\varepsilon^2)}{1 + \varepsilon\left[X_x + X_y y'\right] + O(\varepsilon^2)}$$

$$= y' + \varepsilon\left[Y_x + Y_y y' - (X_x + X_y y')y'\right] + O(\varepsilon^2)$$

Also

$$F(\bar{x},\bar{y}) = F(x + X\varepsilon + O(\varepsilon^2),\ y + Y\varepsilon + O(\varepsilon^2))$$

$$= F(x,y) + (XF_x + YF_y)\varepsilon + O(\varepsilon^2)$$

and if $\frac{d\bar{y}}{d\bar{x}} = F(\bar{x},\bar{y})$ when $\frac{dy}{dx} = F(x,y)$

then to order ε

$$Y_x + (Y_y - X_x)y' - X_y y'^2 = X F_x + Y F_y$$

when $y' = F$ or

$$Y_x + (Y_y - X_x)F - X_y F^2 = X F_x + Y F_y$$

which is Lie's invariance condition

Ex 1 $\quad \dfrac{dy}{dx} = 2y^2 + xy^3$

show Lie's invariance condition is satisfied

by $\quad X = x, \quad Y = -y$

so LIC.

$$0 + (-1-1)F \overset{?}{=} x F_x - y F_y$$

$$-2(2y^2 + xy^3) \overset{?}{=} x(y^3) - y(4y + 3xy^2) \quad ✓$$

Ex 2 $$\frac{dy}{dx} = \frac{x+y+y^2}{xy} = \frac{1}{x} + \frac{1}{y} + \frac{y}{x}$$

LG $$\bar{x} = \frac{x}{1+\varepsilon x}, \quad \bar{y} = \frac{y}{1+\varepsilon x}$$

Inf. $$X = -x^2, \quad Y = -xy$$

so $$Y_x + (Y_y - X_x)F - X_y F^2$$

$$= -y + (-x+2x)\left(\frac{x+y+y^2}{xy}\right)$$

$$= -y + \cancel{x}\left(\frac{x+y+y^2}{\cancel{x}y}\right) = -\cancel{y} + \frac{x}{y} + 1 + \cancel{y} = \frac{x}{y} + 1$$

$$XF_x + YF_y = -x^2\left[-\frac{1}{x^2} - \frac{y}{x^2}\right] - xy\left[-\frac{1}{y^2} + \frac{1}{x}\right]$$

$$= 1 + \cancel{y} + \frac{x}{y} - \cancel{y}$$

$$= 1 + \frac{x}{y} \quad \text{same}$$

so how hard is it to find X & Y given F?