Math 4315 – PDEs

We now move to quasi-linear PDE's
These are of the form

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u)$$

We still use $u_s = u_x x_s + u_y y_s$

CE. $x_s = a,\; y_s = b,\; u_s = c$

these now could be coupled

Ex! $u_x + u u_y = 0$

CE $x_s = 1$ so $u = a(r)$

$y_s = u$ $\quad x = s + b(r)$

$u_s = 0$ $\quad y_s = a(r) \Rightarrow y = a(r)s + c(r)$

So $y = a(r)(x - b(r)) + c(r)$

$$y = a(r)x + c(r) - a(r)b(r)$$

So $y - xu = B(r)$

$u = A(r)$

So $y - xu = f(u)$ or $u = g(y - xu)$

the sol^n is implicit.

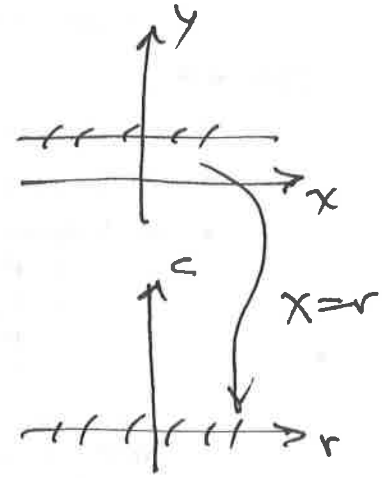
Ex 2 $uu_x + yu_y = u + 2y^2$ $\quad u(x,1) = x+1$

CE $\quad x_s = u \qquad\qquad s=0$

$\qquad\qquad\qquad\quad x = r$

$y_s = y \qquad\qquad y = 1$

$u_s = u + 2y^2 \qquad u = r+1$



1st $y_s = y$ so $y = a(r)e^s$

$s=0, y=1 \Rightarrow a(r) = 1$ so $\boxed{y = e^s}$

Next $\quad u_s = u + 2y^2$

$= u + 2e^{2s}$ ← we need y here!

$$u_s - u = 2e^{2s}$$

integrating factor $\mu = e^{-s}$

so $\frac{\partial}{\partial s}(e^{-s}u) = 2e^{s}$

$$e^{-s}u = 2e^{s} + b(r)$$

Now $s=0$ $u=r+1 \Rightarrow$ $r+1 = 2+b(r)$

$$b(r) = r-1$$

so $u = 2e^{2s} + (r-1)e^{s}$

Now $x_s = u$ * we need u here

$$= 2e^{2s} + (r-1)e^{s}$$

$$x = e^{2s} + (r-1)e^{s} + c(r)$$

$s=0$ $x=r$ $\Rightarrow$ $r = 1+r-1+c \Rightarrow c(r) = 0$

$$x = e^{2s} + (r-1)e^{s}$$

so now we ~~have~~ the sol^n parametrically

$$x = e^{2s} + (r-1)e^{s}$$

$$y = e^{s}$$

$$u = 2e^{2s} + (r-1)e^{s}$$

$$= \underbrace{e^{2s}}_{y^2} + \underbrace{(r-1)e^{s} + e^{2s}}_{x}$$

$$\boxed{u = x + y^2 \quad \text{sol}^n}$$

check $u_x = 1$, $u_y = 2y$

L.S. $u u_x + y u_y = (x+y^2)(1) + y \cdot 2y$

$$= x + y^2 + 2y^2 = x + 3y^2$$

R.S. $u + 2y^2 = x + y^2 + 2y^2 = $ L.S. ✓

$u(x,1) = x + 1$ ✓