Math 4315 - PDEs

Sample Test 2 - Solutions

1. Transform the following PDEs to canonical form. In the case that the type is hyperbolic, transform to both modified and regular form

$$(i) \qquad 6u_{xx} - 5u_{xy} + u_{yy} = 0,$$

$$(ii) \qquad 4u_{xx} + 12u_{xy} + 13u_{yy} = 0,$$

$$(iii) \qquad u_{xx} + 4u_{xy} + 4u_{yy} - u_x = 1,$$

$$(iv) \qquad x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} - xu_x - yu_y = 0,$$

$$(v) \qquad 2y^2u_{xx} - 5xyu_{xy} + 2x^2u_{yy} = 0,$$

$$(vi) \qquad x^2u_{xx} - 2xu_{xy} + 55u_{yy} + xu_x = 0,$$

1(i) The PDE to transform is

$$6u_{xx} - 5u_{xy} + u_{yy} = 0.$$

Here $a = 6$, $b = -5$ and $c = 1$ so $b^2 - 4ac = 1 > 0$ so the PDE is hyperbolic. For the modified hyperbolic form, it is necessary to solve

$$6r_x^2 - 5r_xr_y + r_y^2 = 0, \quad 6s_x^2 - 5s_xs_y + s_y^2 = 0,$$

noting that they factor

$$(2r_x - r_y)(3r_x - r_y) = 0, \quad (2s_x - s_y)(3s_x - s_y) = 0.$$

Thus, solving

$$2r_x - r_y = 0, \quad 3s_x - s_y = 0,$$

gives r and s as

$$r = x + 2y, \qquad s = x + 3y.$$

The derivatives transform as follows:

$$
\begin{aligned}
(i) &\qquad u_x = u_r + u_s, \qquad u_y = 2u_r + 3u_s, \\
(ii) &\qquad u_{xx} = u_{rr} + 2u_{rs} + u_{ss}, \\
(iii) &\qquad u_{xy} = 2u_{rr} + 5u_{rs} + 3u_{ss}, \\
(iv) &\qquad u_{yy} = 4u_{rr} + 12u_{rs} + 9u_{ss},
\end{aligned}
$$

thus transforming the PDE to modified hyperbolic form

$$-u_{rs} = 0, \qquad \Rightarrow \qquad u_{rs} = 0.$$

1(ii). The PDE to transform is

$$4u_{xx} + 12u_{xy} + 13u_{yy} = 0.$$

Here $a = 4$, $b = 12$ and $c = 13$ so $b^2 - 4ac = -64 < 0$ so the PDE is elliptic. Therefore, it is necessary to solve

$$4r_x^2 + 12r_xr_y + 13r_y^2 = 0, \quad 4s_x^2 + 12s_xs_y + 13s_y^2 = 0,$$

using the quadratic formula. Dividing by r_y^2 gives

$$4\left(\frac{r_x}{r_y}\right)^2 + 12\frac{r_x}{r_y} + 13 = 0,$$

and solving gives

$$\frac{r_x}{r_y} = \frac{-12 \pm 8i}{8} = \frac{-3 \pm 2i}{2}.$$

Therefore, we solve the first order PDE

$$r_x = \frac{-3 \pm 2i}{2}\, r_y \quad \text{or} \quad 2r_x - (-3 \pm 2i)\, r_y = 0.$$

Using the method of characteristics gives

$$\frac{dx}{2} = -\frac{dy}{-3 \pm 2i},$$

which leads to

$$\begin{aligned} r, s &= (-3 \pm 2i)\, x + 2y, \\ r, s &= -3x + 2y \pm 2x. \end{aligned}$$

where we choose

$$r = -3x + 2y, \qquad s = 2x.$$

The derivatives transform as follows:

$$\begin{array}{rc} (i) & u_x = -3u_r + 2u_s, \qquad u_y = 2u_r, \\ (ii) & u_{xx} = 9u_{rr} - 12u_{rs} + 4u_{ss}, \\ (iii) & u_{xy} = -6u_{rr} + 4u_{rs}, \\ (iv) & u_{yy} = 4u_{rr}, \end{array}$$

thus transforming the PDE to elliptic form

$$16u_{rr} + 16u_{ss} = 0 \qquad \Rightarrow \qquad u_{rr} + u_{ss} = 0.$$

1(iii). The PDE to transform is

$$u_{xx} + 4u_{xy} + 4u_{yy} - u_x = 1.$$

Here $a = 1$, $b = 4$ and $c = 4$ so $b^2 - 4ac = 0$ so the PDE is parabolic

Therefore, it is necessary to solve

$$r_x + 2r_y = 0,$$

Using the method of characteristics gives

$$\frac{dx}{1} = \frac{dy}{2},$$

which leads to

$$r = 2x - y, \quad s = y,$$

noting that the choice for s is arbitrary. The derivatives transform as follows:

$$\begin{aligned}
&(i) \qquad u_x = 2u_r, \quad u_y = -u_r + u_s, \\
&(ii) \qquad u_{xx} = 4u_{rr}, \\
&(iii) \qquad u_{xy} = -2u_{rr} + 2u_{rs}, \\
&(iv) \qquad u_{yy} = u_{rr} - 2u_{rs} + u_{ss},
\end{aligned}$$

thus transforming the PDE to parabolic form

$$u_{ss} - \frac{u_r}{2} = \frac{1}{4}.$$

1(iv). The PDE to transform is

$$x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} - x u_x - y u_y = 0.$$

Here $a = x^2$, $b = 2xy$ and $c = y^2$ so $b^2 - 4ac = 0$ so the PDE is parabolic Therefore, it is necessary to solve

$$x^2 r_x + xy r_y = 0, \quad \Rightarrow \quad x r_x + y r_y = 0.$$

Using the method of characteristics gives

$$\frac{dx}{x} = \frac{dy}{y}; dr = 0$$

which leads to

$$r = \frac{y}{x}, \qquad s = y,$$

noting that the choice for s is arbitrary. The derivatives transform as follows:

$$\begin{array}{rc} (i) & u_x = -\frac{y}{x^2}u_r, \qquad u_y = \frac{1}{x}u_r + u_s, \\ (ii) & u_{xx} = \frac{y^2}{x^4}u_{rr} + \frac{2y}{x^3}u_r, \\ (iii) & u_{xy} = -\frac{y}{x^3}u_{rr} - \frac{y}{x^2}u_{rs} - \frac{1}{x^2}u_r, \\ (iv) & u_{yy} = \frac{1}{x^2}u_{rr} + \frac{2}{x}u_{rs} + u_{ss}, \end{array}$$

thus transforming the PDE to parabolic form

$$y^2 u_{ss} - y u_s = 0 \quad \Rightarrow \quad u_{ss} - \frac{1}{s}u_s = 0.$$

1(v) The PDE to transform is

$$2y^2 u_{xx} - 5xy u_{xy} + 2x^2 u_{yy} = 0.$$

Here $a = 2y^2$, $b = -5xy$ and $c = 2x^2$ so $b^2 - 4ac = 9x^2y^2 > 0$ for $xy \neq 0$ so the PDE is hyperbolic. For modified hyperbolic form, it is necessary to solve

$$2y^2 r_x^2 - 5xy r_x r_y + 2x^2 r_y^2 = 0, \qquad 2y^2 s_x^2 - 5xy s_x s_y + 2x^2 s_y^2 = 0,$$

noting that they factor

$$(2yr_x - xr_y)(yr_x - 2xr_y) = 0, \quad (2ys_x - xs_y)(ys_x - 2xs_y) = 0.$$

Thus, solving

$$2yr_x - xr_y = 0, \quad ys_x - 2xs_y = 0.$$

gives r and s as

$$r = x^2 + 2y^2, \qquad s = 2x^2 + y^2.$$

The derivatives transform as follows:

$$
\begin{aligned}
&(i) && u_x = 2xu_r + 4xu_s, \qquad u_y = 4yu_r + 2yu_s,\\
&(ii) && u_{xx} = 4x^2u_{rr} + 16x^2u_{rs} + 16x^2u_{ss} + 2u_r + 4u_s,\\
&(iii) && u_{xy} = 8xyu_{rr} + 20xyu_{rs} + 8xyu_{ss},\\
&(iv) && u_{yy} = 16y^2u_{rr} + 16y^2u_{rs} + 4y^2u_{ss} + 4u_r + 2u_s,
\end{aligned}
$$

thus transforming the PDE to modified hyperbolic form

$$u_{rs} - \frac{su_r + ru_s}{(2r-s)(2s-r)} = 0.$$

1(vi). The PDE to transform is

$$x^2u_{xx} - 2xu_{xy} + 5u_{yy} + xu_x = 0.$$

Here $a = x^2$, $b = -2x$ and $c = 5$ so $b^2 - 4ac = -16x^2 < 0$ for $x \neq 0$ so the PDE is elliptic.

Therefore, it is necessary to solve

$$x^2r_x^2 - 2xr_xr_y + 5r_y^2 = 0, \quad x^2s_x^2 - 2xs_xs_y + 5s_y^2 = 0,$$

using the quadratic formula. Dividing by r_y^2 gives

$$x^2\left(\frac{r_x}{r_y}\right)^2 - 2x\frac{r_x}{r_y} + 5 = 0,$$

and solving gives

$$\frac{r_x}{r_y} = \frac{2x \pm 4xi}{2x^2} = \frac{1 \pm 2i}{x}.$$

Therefore, we solve the first order PDE

$$xr_x - (1 \pm 2i)\, r_y = 0.$$

Using the method of characteristics gives

$$\frac{dx}{x} = -\frac{dy}{1 \pm 2i},$$

which leads to

$$\begin{aligned} r, s &= (1 \pm 2i) \ln x + y, \\ r, s &= \ln x + y \pm 2 \ln x. \end{aligned}$$

where we choose

$$r = \ln x + y, \qquad s = 2 \ln x.$$

The derivatives transform as follows:

$$\begin{aligned}
&(i) && u_x = \tfrac{1}{x} u_r + \tfrac{2}{x} u_s, \qquad u_y = u_r, \\
&(ii) && u_{xx} = \tfrac{1}{x^2} u_{rr} + \tfrac{4}{x^2} u_{rs} + \tfrac{4}{x^2} u_{ss} - \tfrac{1}{x^2} u_r - \tfrac{2}{x^2} u_s, \\
&(iii) && u_{xy} = \tfrac{1}{x} u_{rr} + \tfrac{2}{x} u_{rs}, \\
&(iv) && u_{yy} = u_{rr},
\end{aligned}$$

thus transforming the PDE to elliptic form

$$4u_{rr} + 4u_{ss} = 0, \qquad \Rightarrow \qquad u_{rr} + u_{ss} = 0.$$