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Research Article

On obtaining an ε - Compactification of a topological space

Amos Otieno Wanjara

School of Mathematics, Statistics and Actuarial Science, Kaimosi Friends University College, P.O BOX 385-50309, Kaimosi-Kenya.

*Corresponding author's e-mail: awanjara78@gmail.com

Abstract

It is known that one of the most elementary and functional notions of finiteness in analysis, algebra and topology is the notion of compactness yet mathematicians work with many non-compact topological spaces, which have applications that require some properties of compact spaces. The purpose of this paper is to check the conditions under which if *Z* is a compactification of *W*, under what condition can a continuous real-valued function defined on *W* be extended continuously to Z where Z is an ε - compactification of W.

Keywords: Compactification; Equivalent Compactification; the one point Compactificaton; Hausdorff space.

Introduction

One of the most elementary and functional notions of finiteness in analysis, algebra and topology is the notion of compactness. Ennis and Vielma in [1] introduced the concepts of $\alpha - locally\ compact\ \ spaces$ and $\alpha - compactifications$. They showed that every $\alpha -$ T_2 and $\alpha - locally\ compact\ \ space$ has an $\alpha - compactifications$. Max in [2] investigated spaces that have a maximal finite compactification. These are spaces that have compactifications with N-point remainder, but no compactification with N+1-point remainder. In a sense, mathematicians wished to distill out some properties of large spaces by looking at compact spaces that resemble these large spaces. As such, it is in this vein that the concept of compactifying large spaces led to the piecewise development of the Stone-Cech compactification, βX for a topological space *X*.

In [3], Tarizadeh and Rezaee gave new advances on the compactifications of topological spaces, in particular on the Alexandroff and Stone-Cech compactification. In [4], The Stone-Cech compactification is considered to be one of the most important universal properties in topology due to its importance in applications that range from Ramsey theory and topological dynamical systems to computing the dual space of $\ell^{\infty}(\mathbb{R})$, the space of bounded sequence of real numbers.

In mathematical discipline of general topology [4], Stone-Cech compactification is a technique for constructing a universal map from a topological space *X*, to a compact Hausdorff space βX. The Stone-Cech compactification βX of a topological space *X* is the largest compact Hausdorff space generated by *X*, in the sense that any map from *X* to a compact Hausdorff space factors through βX(in a unique way).

Munkress and James [5] defines a compactification of a topological space X as a compact Hausdorff space *Y* containing X as a subspace such that $\bar{X} = Y$. In order to move past the naïve intuition behind the historical notion on finiteness, the mathematical community required the leadership and direction of mathematicians such as Bernhard Riemann, John Von Neumann and Marshall Stone. In [6], Riemann provided the first example of a compactification using the intuitively infinite and non- compact topological space $\mathbb{C}$ with the construction of the Riemann sphere. The Riemann sphere is what is known as the one-point compactification of the complex plane $\mathbb{C}$.

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Research methodology

Definition 1.1 [7, Definition 2.6]

All subspaces of compact Hausdorff spaces are completely regular Hausdorff spaces as those that can be compactified. Such spaces are now called Tychonoff spaces. In topology and related branches of mathematics, Tychonoff spaces and completely regular spaces are kinds of topological spaces.

Definition 1.2 [8, Definition 1.0]

A metric space X is said to be compact if every sequence in X has a convergent subsequence. A subset M of X is said to be compact if M is compact considered as a subspace of X, that is every sequence in M has a convergent sequence whose limit is an element of M.

Definition1.3 [9, Definition 2.2]

A space X is said to be locally compact at x if there is some compact subset C of X that contains a neighborhood of x. If X is locally compact at each of its points, X is said to be locally compact.

Definition 1.4 [10, Definition 3.0]

A topological space X is called a Hausdorff space if for each pair, x_1, x_2 of distinct points of X, there exists neighborhood U_1 *and* U_2 of x_1 *and* x_2 respectively that are disjoint.

Lemma 1.5 [11, Lemma 1.0]

X is a normal topological space if and only if whenever A,B are disjoint closed sets in X then there exists a continuous function $f: X \to [0,1]: f(a) = 0, f(b) = 1\ \forall\ a \in A, b \in B$.

Theorem 1.6 [12, theorem 1.1]

An arbitrary product of compact spaces is compact in the product topology.

Theorem 1.7 [13, theorem 2.0]

The completely regular spaces are precisely those spaces which can be embedded in a product of copies of the closed unit interval $I = [0,1]$

Remark 1.7.1

As compact Hausdorff spaces are normal spaces [6], combining Urysohn's lemma with Tychonoff's theorem for completely regular spaces gives us the following crucial nugget of topological information:

'No larger class of topology spaces can be studied by means of embeddings into compact Hausdorff spaces.'

This means that we need only to look at completely regular spaces when defining our desired compactification.

Definition 1.8 [14, Definition 3.3]

Let X and Y be topological spaces. Let $f: X \to Y$ be a bijection.If both the functions f and the inverse function $f^{-1}: Y \to X$ are continuous, then f is called a homeomorphism.

Definition 1.9

A filter on a set X is a collection $\mathcal{F}$ of subsets of X satisfying:

i. $X \in \mathcal{F}$ *but* $\emptyset \notin \mathcal{F}$

ii. If $X \in \mathcal{F}$ and $A \subset B \subset X$ then$B \in \mathcal{F}$

iii. A finite intersection of sets in $\mathcal{F}$ is in $\mathcal{F}: if\ A_{1,2} \in \mathcal{F}, then\ A_1 \cap A_2 \in \mathcal{F}$

Definition 2.0

An ultrafilter on a set X is a filter $\mathcal{F}$ on X which is maximal with respect to inclusion i.e it is a filter $\mathcal{F}$ for which any other filter $\mathcal{F}'$ on X satisfying $\mathcal{F}'$ on X satisfying $\mathcal{F}' \supset \mathcal{F}$actually satisfies $\mathcal{F}' = \mathcal{F}$.Every Principal filter is an ultrafilter.

Definition2.1 [15, Definition 1.1]

Let X and Y be topological spaces. The product topology on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form, $U \times V$ where U is an open subset of X and V is an open subset of Y

Definition 2.2

Suppose that one-point sets are closed in X. Then X is said to be regular if for each pair consisting of a point x and a closed set B disjoint from x, there exists disjoint open sets containing x and B respectively.

Definition 2.3

A space X is completely regular if one point sets are closed in X and if for each point x_0 and each closed set A not containing x_0, there is a continuous function $f: X \to [0,1]$ such that $f(x_0)=1$ and $f(A)=\{0\}$

Definition 2.4

A compactification of a space X is a compact Hausdorff space Y containing X such that X is dense in Y (that is $\bar{X} = Y$).Two compactificationsY_1 and Y_2 of X are said to be equivalent if there is a homeomorphism $h: Y_1 \rightarrow Y_2$ such that h(x)= x $\forall\ x \in X$

Results and discussions

In this section, we give the results. We begin with the following remark.

Remark 3.0.0:

In order for W to have a compactification, W must be completely regular. Conversely, every completely regular space has at least one compactification.'If Z is a compactification of W, under what condition can a continuous real-valued function ε $\boldsymbol{f}$ defined on W be extended continuously to Z? We express the answer to this question in the following theorems:

Theorem 3.0.1

Let W be completely regular; let $\varepsilon(W)$ be its compactification. Then every bounded continuous real-valued function on W can be uniquely extended to a continuous real-valued function on $\varepsilon(W)$.

Proof

The compactification $\varepsilon(W)$ is induced by the imbedding $r: W \rightarrow \prod I_\delta$(defined $r: W \rightarrow \prod_{\delta \in J} I_\delta$by the rule r(w)=$(\varepsilon f_\delta(w), \delta \in J)$.This means that there is an imbedding $R: \varepsilon(W) \rightarrow \prod I_\delta$ that equals r when restricted to the subspace W of $\varepsilon(W)$. Given a continuous bounded real-valued function on W it equals $f_\varepsilon \forall \varepsilon \in J$. Now if $\pi_\varepsilon: \prod I_\delta \rightarrow I_\varepsilon$ is projection onto the ε^{th} coordinate then the composite map $\pi\varepsilon^0 R: \varepsilon(W) \rightarrow I_\varepsilon$ is the desired extension of f_ε for if $w \in W$, we have

$$\pi_\varepsilon(R(w)) = \pi_\varepsilon(r(w)) = \pi_\varepsilon((\varepsilon f_\delta(w)_{\delta \in J}) = f_\varepsilon(w)$$

Theorem 3.0.2

Let W be completely regular. Let $Z_1 and\ Z_2$ be two compactifications of W having the extension property. Then there is a homeormorphism $\vartheta \approx \varepsilon$ of $Z_1 onto\ Z_2$ such that $\vartheta(w) = w\ \forall\ w \in W$.

Proof:

Case 1: Suppose Z is a compactification of W having the extension property. If M is any compact Hausdorff space and n: ε $W \rightarrow M$ is any continuous function, then n can be extended to a continuous function t mapping σ into M. To prove this fact, note that M is completely regular so that it can be imbedded in $[0,1]^J\ \forall J$. So we may as well assume that $M \subset [0,1]^J$. Now consider $n: \varepsilon W \rightarrow M \subset [0,1]^J \subset \mathbb{R}^J$. Each component function n_δ of the map n is a continuous bounded real-valued function on ε W ; by hypothesis, n_δ can be extended to a continuous map t_δ of Z into $\mathbb{R}$. Define $t: Z \rightarrow \mathbb{R}^J$ by setting $t(z)=t_\delta(z)_{\delta \in J}$. The map t is continuous because $\mathbb{R}^J$ has the product topology. We assert that t actually maps Z into the subspace M. For $n(w)$ is contained in M and $t(\varepsilon\ W)= n(\varepsilon\ W)$. Since M is closed in $\mathbb{R}^J$, it follows that $\overline{J(\varepsilon W)} \subset M$. By continuity of t, $t(z)$ = $t(\varepsilon \bar{W}) \subset \overline{t(\varepsilon W)}$.Therefore t maps Z into M.

Case 2: Consider the inclusion mapping $t_2: \varepsilon W \rightarrow Z_2$. It is a continuous map of ε W into the compact Hausdorff space Z_2. Because Z_1. Has the extension property, we may by step 1, extend t_2 to a continuous map $\varepsilon f_2: Z_1 \rightarrow Z_2$. Similarly, we may extend the inclusion map $t_1: \varepsilon W \rightarrow Z_1$ to a continuous map $\varepsilon f_1: Z_1 \rightarrow Z_2$. The composite $\varepsilon f_{1o2}: Z_1 \rightarrow Z_2$ has the property that $\forall \in \varepsilon W$, one has $\varepsilon f_1(f_2(w)) = w$. Therefore εf_{1o2} is a continuous extension of the identity map $i_w: \varepsilon W \rightarrow \varepsilon W$. But the identity map of Z_1 is also a continuous extension of i_w. By uniqueness of extensions, εf_{1o2} must equal the identity map of Z_1. Similarly εf_{1o2} must equal the identity map of Z_1. Thus εf_1 and εf_2 are homeomorphisms.

Conclusions

The concept of compactification in analysis, algebra and topology has been studied over a period of time. Various ways in which a compactification can be constructed include the use of: $C^*algebra$s, Products, ultrafilters and even natural numbers. 'If Z is a compactification of W, under what condition can a continuous real-valued function ε $\boldsymbol{f}$ defined on W be extended continuously to Z? In this paper we have shown that there exists an ε -compactification Z on a Topological space provided W is completely regular.

Conflicts of interest

Authors declare no conflict of interest.

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