Math 1497 - Calc 2

Tests

(1) nth Term Test

Given $\sum_{n=1}^{\infty} a_n$ if $\lim_{n\to\infty} a_n = \#$ (not 0)

the series diverges.

Ex 1 $\sum_{n=1}^{\infty} \frac{2n^2+1}{n^2-2n+3}$ $\lim_{n\to\infty} \frac{2n^2+1}{n^2-2n+3} \overset{LH}{=} \lim_{n\to\infty} \frac{4n}{2n-2}$

LH $\lim_{n\to\infty} \frac{4}{2} \neq 0$ so the series div by the nth term test.

2) Integral Test

If $f(n) = a_n$ if f is (i) >0 (ii) cont's (iii) dec

we can apply the test

if $\int_1^{\infty} f(n)\,dn$ conv (div) $\sum_{n=1}^{\infty} a_n$ conv (div)

Ex2 $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$

$f(n) = \frac{1}{1+n^2}$ & f is cont's ✓ >0 ✓ $f' = \frac{-2n}{(1+n^2)^2} < 0$
so dec ✓

so test applies

$$\int_1^{\infty} \frac{dn}{1+n^2} = \lim_{b\to\infty} \int_1^b \frac{dn}{1+n^2} = \lim_{b\to\infty} \tan^{-1} n \Big|_1^b$$

$$= \lim_{b\to\infty} \tan^{-1} b - \tan^{-1} 1 = \frac{\pi}{2} - \pi/4 = \frac{\pi}{4}$$

so by the $\int$ test, the series converges.

Let us consider this series a little more

$\sum_{n=1}^{\infty} \frac{1}{1+n^2}$. Let us look at the term way out

Almost like

$a_{10} = \frac{1}{1+10^2} = \frac{1}{101} = .0099099$ $\qquad \frac{1}{100} = .01$

$a_{100} = \frac{1}{1+100^2} = \frac{1}{10001} = .00009999$ $\qquad \frac{1}{10000} = .0001$

$a_{1000} = \frac{1}{1+1000^2} = \frac{1}{1000001} = .0000009999$ $\qquad \frac{1}{1000000} = .000001$

what I'm saying is the series

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} \quad \text{looks like} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges by $\int$ test $\qquad$ $p=2$ converges

This is the 3^{rd} test – Limit Comparison Test (LCT)

\# 3 LCT

Given $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$

if $\lim_{n\to\infty} \frac{a_n}{b_n} = \#$ (not 0)

both series do the same. (conv or div)

Previous Ex

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} \quad \text{compare w/} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{1+n^2}} = \lim_{n\to\infty} \frac{1+n^2}{n^2} = 1$$

since $\sum \frac{1}{n^2}$ conv

then $\sum \frac{1}{1+n^2}$ conv by LCT.

Ex 2 Consider $\sum_{n=1}^{\infty} \frac{n+1}{2n^2+1}$

compare w/ $\sum \frac{1}{n}$ harmonic div

$$\lim_{n\to\infty} \frac{\frac{n+1}{2n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2+n}{2n^2+1} = \frac{1}{2} \quad \left(\begin{array}{c}\#\text{ not}\\ \text{zero}\end{array}\right)$$

so by LCT our series diverges

Ex 3 $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$

compare with $\sum \frac{1}{2^n}$ ~~ha~~ geometric $r=\frac{1}{2}$

$$\lim_{n\to\infty} \frac{\frac{1}{2^n}}{\frac{1}{2^n+1}} = \lim_{n\to\infty} \frac{2^n+1}{2^n} = \lim_{n\to\infty} 1+\frac{1}{2^n} = 1$$

so our series converges by LCT

could also have done $\lim_{n\to\infty} \frac{\frac{1}{2^n+1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n+1}$ LH

Ex 4 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+3^n}$

Compare w/ $\sum \frac{1}{2^n}$ or $\sum \frac{1}{3^n}$

this one or this one

$$\lim_{n\to\infty} \frac{\frac{1}{3^n}}{\frac{1}{2^n+3^n}} = \lim_{n\to\infty} \frac{2^n+3^n}{3^n} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n + 1 \to 1$$

so $\because$ $\sum \frac{1}{3^n}$ conv. then by LCT $\sum \frac{1}{2^n+3^n}$ conv

Ex 5 $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Not obvious but compare with $\sum \frac{1}{n}$ ~~forever~~

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \text{"}\frac{0}{0}\text{"} \ \text{LH} \ \lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right)\left(-\frac{1}{\cancel{n^2}}\right)}{-\frac{1}{\cancel{n^2}}} = 1$$

So by LCT our series div.