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Research Article

Spectral Characterization of Convexoid Operators

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Abstract

Spectrum is a very useful property in studying operators on Banach spaces particularly, Hilbert spaces. In particular, the geometrical properties of spectrum often provide useful information about algebraic and analytic properties of an operator. The theory of spectrum played a crucial role in the study of some algebraic structures especially in the associative context. The spectrum of an operator depends strongly upon the base of scalars. Motivated by theoretical study and applications, researchers have considered different generalizations of spectrum. This study gives results of spectrum of convexoid operators. Let *H* be an infinite dimensional complex Hilbert space and *B(H)* be algebra of all bounded linear operators on *H*. $T \epsilon B(H)$ is said to be convexoid if the closure of the numerical range coincides with the convex hull of its spectrum. In this paper, the study determines the spectrum of convexoid operators. This work considers some results on spectra due to Rota, Hildebrandant, Furuta and Nakamoto among others.

Keywords: Resolvent set; Convexoid operator; Spectral radius; Spectrum of operator.

Introduction

Spectral theory of linear operators plays a central role in modern Mathematics [1]. Generally spectral theorem is greatly applied in almost any physical problem which can be formulated in terms of a linear operator, the spectrum of the operator is a quantity of basic physical interest [2]. For a general linear operator it is often hard to obtain information even of a qualitative nature about the spectrum , but for a self-adjoint operator on Hilbert space this is a task of much less difficulty. Spectral theorem then lies at the heart of any discussion of such equations. In investigating the structure of the spectrum of an operator in question, it is known that [3], the spectral space of any class of operators includes properly, the spectral space of all operators from its subclasses.

It has been noted in [4] that the spectrum of a self adjoint operator lies along the real line, that of the unitary lies on the unit circle, that of a projection consists of the points 0 and 1, and that of a normal operator can be any compact set in the complex plane. For more information about spectrum see [5] and [6] and the references there in. The spectrum of T denoted by $((T) = \lambda : T - \lambda I$ is not invertible.) Since $T - \lambda I$ is not invertible then $\sigma(T)$ can be split into many disjoint parts. The classical partitioning comprises three parts as follows [7] and [8]. The point spectrum of T consists of all $\lambda \in \sigma(T)$ such that $T - \lambda I$ is not one to one (injective). Denoted by $\sigma_P(T)$. The continuous spectrum of T consists of all $\lambda \epsilon \sigma(T)$ such that $T - \lambda I$ is one to one but not onto and $ran(T - \lambda I)$ is dense in H. Denoted by $\sigma_C(T)$. The residual spectrum of T consists of all $\lambda \in \sigma(T)$ such that $(T - \lambda I)$ is one to one but not onto and $ran(T - \lambda I)$ is not dense in H. It is denoted by $\sigma_R(T)$.

Being pairwise disjoint, then $\sigma(T) = \sigma_P(T) \cup \sigma_C(T) \cup \sigma_R(T)$. Residual spectrum is further split into two disjoint parts $\sigma_R(T) = \sigma_{R1}(T) \cup \sigma_{R2}(T)$. The point spectrum is split into four disjoint parts [9], $\sigma_P(T) = \cup_{i=1}^{4} \sigma_{P_i}(T)$. There are some overlapping parts of the spectrum which are commonly used. They include compression spectrum $\sigma_{CP}(T)$ [10]. Approximate point spectrum $\sigma_{AP}(T)$. The set of all convexoid operators on H is denoted by $CX(H)$. In this section, definitions of some terms that are useful in our work are given.

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Definition 1. [11]. Resolvent set of an operator $T \in B(H)$, denoted by $\rho(T)$ is set of complex numbers λ such that $(T-\lambda I): H \to H$ is one to one and onto.

Definition 2. [12]. Spectrum of T denoted by $\sigma(T) = \{\lambda : T - \lambda I \text{ is not invertible}\}$ is the complement of the resolvent set.

Definition 3. [13] Spectral radius $r(T)$ of an operator T on H is given by : $r(T) = \sup\{|\lambda| \lambda \in \sigma(T)\}$.

Definition 4. [14] Convexoid operator, $CX(H)$, is a bounded liner operator T on a complex Hilbert space H such that the closure of the numerical range coincides with the convex hull of its spectrum i.e $\overline{W(T)} = co\, \sigma(T)$.

Research methodology

Applications of two main techniques i.e. Laurent series and uniform boundedness principle were used as in formulation of the results. For a successful completion of this work, background knowledge of topology, functional analysis, the operator theory, especially normal operators, self-adjoint operators, hyponormal operators on a Hilbert space, numerical range and the spectrum of operators on a Hilbert space is vital. Some known fundamental principles which shall be useful in our research have been stated. The methodology involved the use of known inequalities and techniques like Cauchy - Schwartz inequality, Minkowski's inequality, parallelogram law and the polarization identity. Lastly, the study has used the technical approach of tensor products in solving the stated problem. Also the methodology involved the use of known inequalities and techniques like the polarization identity [7].

This study has also used the technical approach of derivatives of fuzzy sets to characterize the numerical range of convexoid operators. Spectral theory of linear operators on Hilbert spaces is a pillar in several developments in mathematics, physics and quantum mechanics. Its concepts like the spectrum of a linear operator, eigenvalues and vectors, spectral radius, spectral integrals among others have useful applications in quantum mechanics, a reason why there is a lot of current research on these concepts and their generalizations. Spectral theory is described as a rich and important theory as it relates perfectly with other areas including measure and integration theory and theory of analytic functions. For this particular work, Schwarz inequality [8], polarization identity, inner product [5] and parallelogram law [10] were used as in the results.

Results and discussion

In this section, results on spectrum of convexoid operators are given. The work begins by discussing spectrum of bounded operator $T \in CX(H)$ denoted by $\sigma(T)$ and give some of its properties. Properties of the spectrum are as below [13]

Remark 5. If $\in CX(H)$, then:

(i) $\sigma(T)$ is non-void.

(ii) $\sigma(T)$ is closed in $(\mathbb{C}, d)$. (Where $(\mathbb{C}$,d) is metric space with metric d).

(iii) $\sigma(T) \subseteq \bar{\mathbb{N}}$ $(0,\|T\|)$, (Where $\bar{\mathbb{N}}$ $(0,\|T\|)$ is closed neighborhood of O with radius $\|T\|$.)

Proposition 6. Let H be a real Hilbert space and $T \in CX(H)$ be self adjoint. Then $\sigma(T) \neq \phi$ and there is a $\lambda \in \sigma(T)$ such that $|\lambda| = r(T)$.

Proof: For a self adjoint operator T, $\|T\| = \sup\{|\langle Tx, x\rangle| : x \in H$ and $\|x\| = 1\}$. Then there is a sequence (x_n) of elements of H such that $\|x_n\| = 1, \forall n \in \mathbb{N}$, and $\langle Tx_n, x_n\rangle \to \|T\|$ or $\langle Tx_n, x_n\rangle \to -\|T\|$. In the first case , it follows that

$$\|(\|T\|I - T)x_n\|^2 = \|T\|^2\|x_n\|^2 - 2\|T\|\langle Tx_n, x_n\rangle + \|Tx_n\|^2$$
$$\leq \|T\|^2 - 2\|T\|\langle Tx_n, x_n\rangle + \|T\|^2 \to 0 \text{ as } n \to \infty.$$

In the second case $\|(\|T\|I + T)x_n\|^2 \to 0$ as $n \to \infty$. Consequently, $\|T\| \in \sigma(T)$ in the first case and $-\|T\| \in \sigma(T)$ in the second case. Moreover, $\sigma(T) \neq \emptyset$. ∎

The next proposal compares the spectral radius to the norm of a convexoid operator.

Proposition 7. For any operator $T \in CX(H), r(T) \leq \|T\|$.

Proof:

$$r(T) = \inf\{\|T^n\|^{\frac{1}{n}} : n \in \mathbb{N}\}$$
$$= \lim_{n\to\infty}\{\|T^n\|^{\frac{1}{n}}\}$$
$$\leq \|T\|.$$

Therefore $r(T) \leq \|T\|$. Thus $\sigma(T) \subseteq \mathbb{N}(0, \|T\|)$. ∎

For the spectral radius we have the spectral radius formula and its proof.

Theorem 8. Let $CX(H)$ be a unital Banach algebra and let $x \in CX(H)$. Then the limit $\lim_{n\to\infty}\|x^n\|^{\frac{1}{n}}$ Exists and satisfies $\lim_{n\to\infty}\|x^n\|^{\frac{1}{n}} = r_T(x) = inf_n\|x^n\|^{\frac{1}{n}}$. Moreover, $\alpha, \beta \in \sigma(T)$ for any $\alpha, \beta \in \mathbb{C}$.

Proof:

$\gamma(x) = \inf\{\|x^n\|^{\frac{1}{n}} : n = 1, 2, \ldots\}$. We shall show that $\|x^n\|^{\frac{1}{n}} \to \gamma(x)$ as $n \to \infty$. Given that any $\varepsilon > 0$, let $k \in \mathbb{N}$ be such that $\|x^k\|^{\frac{1}{k}} < \gamma(x) + \varepsilon$. For any $n \in \mathbb{N}$ write n as $n = \alpha k + \beta$ where $0 \le \beta < k$ and $\alpha, \beta \in \mathbb{Z}^+$. (Note that $k$is fixed and α, β depend on n.) Then $\frac{\beta}{n} \to 0$ as $n \to \infty$ since β is always less than k. Also $1 = \frac{n}{n} = \frac{\alpha k+\beta}{n} = \frac{\alpha k}{n} + \frac{\beta}{n}$ so $\frac{\alpha k}{n} \to 1$ as $n \to \infty$ that is , $\frac{\alpha}{n} \to \frac{1}{k}$ as $n \to \infty$. Now,

$$\|x^n\|^{\frac{1}{n}} = \left\|(x^k)^\alpha x^\beta\right\|^{\frac{1}{n}}$$
$$\le \left\|x^k\right\|^{\frac{\alpha}{n}}\|x\|^{\frac{\beta}{n}}$$

And the right hand side converges to $\left\|x^k\right\|^{\frac{1}{k}}$ as $n \to \infty$ which is less than $\gamma(x) + \varepsilon$. Hence for all sufficiently large n,

$$\|x^n\|^{\frac{1}{n}} \le \|x^n\|^{\frac{\alpha}{n}}\|x\|^{\frac{\beta}{n}}$$
$$< \gamma(x) + \varepsilon.$$

Conversely, $\gamma(x) \le \|x^n\|^{\frac{1}{n}}$ for any $n = 1, 2, \ldots$ and so $\gamma(x) \le \|x^n\|^{\frac{1}{n}} < \gamma(x) + \varepsilon$ for all sufficiently large n. Thus $\|x^n\|^{\frac{1}{n}}$ converges to $\gamma(x)$ as $n \to \infty$ i.e $\lim_n\|x^n\|^{\frac{1}{n}}$ exists and is equal to $inf_m\|x^m\|^{\frac{1}{m}}$. Next it must now be shown that the above limit is equal to $r_T(x)$. Recall, first that for any $y \in T, \sigma_T(y) \subseteq \{\lambda : |\lambda| \le \|y\|\}$ and so $r_T(y) \le \|y\|$. It follows that $\{\lambda^n : \lambda \in \sigma_T(x)\} = \sigma_T(x^n)$ and so $r_T(x^n = r_T(x)^n)$, for any $n \in \mathbb{N}$. But $r_T(x^n) \le \|x^n\|$ and so we obtain

$$r_T(x)^n = r_T(x^n) \le \|x^n\|$$
$$\Rightarrow r_T(x) \le \|x^n\|^{\frac{1}{n}} \text{ for all } n$$
$$\Rightarrow r_T(x) \le \gamma(x).$$

Next it is shown that $r_T(x) \ge \gamma(x)$. Let $\varphi \in T^*$. Then $g : \lambda \to \varphi((x - \lambda I)^{-1})$ is analytical on $\frac{\mathbb{C}}{\sigma_T(x)}$ and so has a Laurent series expansion on $\{\lambda : |\lambda| > r_A(x)\}$. But for $|\lambda| > \|x\|$, it is known that g has the expansion $g(\lambda) = \sum_{n=0}^{\infty}\frac{\varphi(x^n)}{\lambda^{n+1}}$.

This must therefore be absolutely convergent in the region $\{\lambda : |\lambda| > r_T(x)\}$. Fix λ with $|\lambda| > r_T(x)$. Then, in particular, $\varphi\frac{(x^n)}{\lambda^{n+1}} \to 0$ as $n \to \infty$. This holds for any $\varphi \in T^*$ and so by the uniform boundedness principle, it follows that $\left(\frac{x^n}{\lambda^{n+1}}\right)$ is a bounded sequence in A, that is there is $k > 0$ such that $\left\|\frac{x^n}{\lambda^{n+1}}\right\| \le k$ for all n. Hence $\|x^n\| \le k|\lambda||\lambda^n|$ and so $\|x^n\|^{\frac{1}{n}} \le (k|\lambda|)^{\frac{1}{n}}|\lambda|$ letting $n \to \infty$ gives $\gamma(x) \le |\lambda|$. This holds for any λ with $|\lambda| > r_T(x)$ and so we deduce that $\gamma(x) \le |\lambda|$. This holds for any λ with $|\lambda| > r_T(x)$ and so we deduce that $\gamma(x) \le r_T(x)$. It follows that $r_T(x) = \gamma(x)$. ∎

Conclusions

The spectrum is a closed disc as seen in this work. It would be interesting to determine the spectrum of other operators that have not been studied. Certain properties of convexoid operators have been characterized for example continuity and linearity but spectrum and spectra of posinormal operators have not been considered. Also the relationship between the numerical range and spectrum has not been determined for convexoid operators. Therefore the objectives of this study were: to investigate numerical ranges of convexoid operator, to investigate the spectra of convexoid operators and to establish the relationship between the numerical range and spectrum of a convexoid operator. The methodology involved use of known inequalities like Cauchy-Schwartz inequality, Minkowski's inequality, the parallelogram law and the polarization identity to determine the numerical range and spectrum of convexoid operators and our technical approach shall involve use of tensor products. The results obtained shall be used in classification of Hilbert space operators and shall be applied in other fields like quantum information theory to optimize minimal output entropy of quantum channel, to detect entanglement using positive maps and for local distinguishability of unitary operators. The numerical range of a bounded convexoid operator A acting on a complex Hilbert space H is an ellipse whose foci are the eigenvalues of A.

Conflicts of Interest

Authors declare no conflict of interest.

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