Math 4381 / 6378 Symmetry

Last class we found the symmetries of the heat eq^n

$$u_t = u_{xx}$$

and found them to be

$$T = c_1 t^2 + c_2 t + c_3$$

$$Q_t = Q_{xx}$$

$$X = \left(c_1 t + \frac{c_2}{2}\right)x + c_4 t + c_5$$

$$U = \left(-\frac{1}{4}c_1 x^2 - \frac{c_4}{2}x - \frac{1}{2}c_1 t + c_6\right)u + Q(t,x)$$

Now we wish to determine whether any of these symmetries can be used to actually solve an IVP. For example

$$u(x,0) = \sin x$$

we actually know that

$$u(x,t) = e^{-t}\sin x \quad \text{is the sol}^n$$

we substitute

$t=0 \quad u=\sin x \quad u_x=\cos x \quad u_t=u_{xx}=-\sin x$

into the invariant surface condition w/

T & X & U given on 1st page so

set $Q=0$

$$-c_3\sin x+\left(\frac{c_2}{2}x+c_5\right)\cos x=\left(-\frac{c_1}{4}x^2-\frac{c_4}{2}x+c_6\right)\sin x$$

this must be satisfied for all x

so $c_2=c_5$ right away

also $c_1=c_4=0$

leaving

$$-c_3\sin x=c_6\sin x$$

so if $c_6=-c_3$ then ~~HAS~~ ISC is satisfied

so $\quad c_3u_t=-c_3u \quad$ a $\quad u_t+u=0 \Rightarrow u=e^{-t}F(x)$

$u_t=u_{xx} \Rightarrow \quad -e^{-t}F=e^{-t}F'' \Rightarrow F''+F=0$

so $F = f_1 \sin x + f_2 \cos x$

sol^n $u = e^{-t}(f_1 \sin x + f_2 \cos x)$

Now $u(x,0) = \sin x \Rightarrow f_1 = 1$ & $f_2 = 0$

giving $u = e^{-t} \sin x$

<u>Note</u>: if $u(x,0) = e^{-t} F(x)$
& $u(x,0) = \sin x$ $\Big\} \Rightarrow F(x) = \sin x$ right away;

Ex2 $u_t = u_{xx}$ $u(x,0) = e^{-x^2}$

so $u_x = -2x e^{-x^2}$ $u_t = u_{xx} = (4x^2-2)e^{-x^2}$

sub $t=0$, u, u_t, u_x into ISC

so

$c_3(4x^2-2)\cancel{e^{-x^2}} + \left(\frac{c_2}{2}x + c_5\right)(-2x\cancel{e^{-x^2}})$

$= \left(-\frac{c_1}{4}x^2 - \frac{c_4 x}{2} + c_6\right)\cancel{e^{-x^2}}$

Now compare terms involving x

$1)\quad -2c_3 = c_6$

$x)\quad -2c_5 = -\frac{c_4}{2}$

$x^2)\quad 4c_3 - c_2 = -\frac{c_1}{4}$

$c_6 = -2c_3 \qquad c_4 = 4c_5 \qquad c_1 = 4c_2 - 16c_3$

So we have 3 free parameters

Let's choose $c_5 = 1$ rest $= 0$

so $T = 0 \quad X = 4t+1, \quad \sigma = -xu$

ISC $\quad (4t+1)\, u_x = -xu$

$$u = F(t)\, e^{-\frac{x^2}{4t+1}}$$

Sub into $u_t = u_{xx}$

$$4tF' + F' + 2F = 0 \Rightarrow F(t) = \frac{c}{\sqrt{4t+1}}$$

sol^n $\quad u = \dfrac{c\, e^{-\frac{x^2}{4t+1}}}{\sqrt{4t+1}}$

$u(x,0) = e^{-x^2} \Rightarrow c = 1$

sol^n $\quad u = \dfrac{e^{-\frac{x^2}{4t+1}}}{\sqrt{4t+1}}$